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An Updated Set of Nonlinear Eigenvalue Problems

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Abstract

This is the description of the new changes made in version 4.0 of the NLEVP MATLAB toolbox. The collection and its organization are described in a separate paper. A user's guide describes how to install and use the NLEVP MATLAB toolbox.

1 Introduction

NLEVP is a MATLAB toolbox, which provides a collection of nonlinear eigenvalue problems. For details of the design and organization of the collection, and a description of the problems in version 3.0, see [7]. The NLEVP toolbox is available on the GITHUB repository at `https://github.com/ftisseur/nlevp`. The MATLAB codes are also available at MATLAB Central File Exchange. For details of how to install and use the toolbox see [6].

This document describes the changes made in version 4.0 of NLEVP, which include the addition of three new problem properties, two new outputs, and 22 new problems (see Table 3).

A convenient general form for expressing the matrix-valued function $F : \mathbb{C} \to \mathbb{C}^{m\times n}$ defining a nonlinear eigenvalue problem (NEP) $F(\lambda)x = 0$, is

$$F(\lambda) = \sum_{i=0}^{k} f_i(\lambda)A_i, \tag{1}$$

where the $f_i : \mathbb{C} \to \mathbb{C}$ are nonlinear functions and $A_i \in \mathbb{C}^{m\times n}$. Particular cases include

- quadratic eigenvalue problems (QEPs) corresponding to $k = 2$ and $f_i(\lambda) = \lambda^i$, $i = 0, 1, 2$,
- polynomial eigenvalue problems (PEPs) corresponding to $f_i(\lambda) = \lambda^i$, $i = 0, \dots, k$,
- rational eigenvalue problems (REPs), where the $f_i(\lambda)$ are rational functions of λ.

We give in Table 1 a list of identifiers for the types of problems available in the collection and in Table 2 a list of identifiers that specify the properties of problems in the collection. These properties can be used to extract specialized subsets of the collection for use in numerical experiments. We refer to [7, Sec. 2] for their definition. The identifiers `banded`, `tridiagonal`, and `low-rank` have been added to version 4.0 of NLEVP. The property `banded` is given to problems with coefficient matrices $A_i \in \mathbb{C}^{n\times n}$ having bandwidth less than $n/5$. The property `low-rank` is given to problems with coefficient matrices A_i of low-rank.

We have also added two new outputs to the `nlevp` function:

- a third output, called `F`, that returns the function handle for the matrix-valued function F in (1). The function handle `F` should be used in place of

  ```
  >> F = @(lam) nlevp('eval',name,lam)
  ```

  whose evaluation is slower for large size problems than that of the new third output;
- a fourth output, called `xcoeffs`, that returns a $2\times(k+1)$ cell array such that if ith coefficient matrix $A_i = E_iF_i$ for some rectangular matrices E_i, F_i of low rank then E_i is given by `xcoeffs{1,i}` and F_i is given by `xcoeffs{2,i}`. This fourth output is only returned for problems with the `low-rank` property.

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Table 1: Problems available in the collection and their identifiers.

qep	quadratic eigenvalue problem
pep	polynomial eigenvalue problem
rep	rational eigenvalue problem
nep	other nonlinear eigenvalue problem

Table 2: List of identifiers for the problem properties.

nonregular	symmetric	hyperbolic
real	hermitian	elliptic
nonsquare	T-even	overdamped
sparse	*-even	proportionally-damped
scalable	T-odd	gyroscopic
parameter-dependent	*-odd	low-rank
solution	T-palindromic	
random	*-palindromic	
tridiagonal	T-anti-palindromic	
banded	*-anti-palindromic	

Table 3: New problems in NLEVP version 4.0.

Quadratic	bcc_traffic	circular_piston	damped_gyro
	deformed_consensus	disk_brake100	disk_brake4669
	elastic_deform	utrecht1331	
Rational	photonic_crystal	railtrack_rep	railtrack2_rep
Nonlinear	bent_beam	bucking_plate	canyon_particle
	distributed_delay1	nep1	nep2
	pdde_symmetric	pillbox_cavity	pillbox_small
	sandwich_beam	time_delay2	

2 Collection of new problems

This section contains a brief description of the new problems added to version 4.0 of the collection. The identifiers for the problem properties are listed inside curly brackets after the name of each problem. The new problems are listed in Table 3. All the problems are summarized in Table 4 and Table 5.

Bcc_traffic {`pep,qep,real,parameter_dependent,scalable,sparse,tridiagonal,banded`}. This $n \times n$ monic quadratic eigenvalue problem $Q(\lambda) = \lambda^2 I + \lambda D + K$ arises in the stability analysis of non-identical vehicles under bilateral cruise control (BCC) [31]. The $n \times n$ nonsymmetric tridiagonal matrices K and D are defined by

$$D = K_v S, \qquad K = K_d S, \qquad S = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix},$$

where both $K_v = \mathrm{diag}(k_v^{(1)}, \ldots, k_v^{(n)})$ and $K_d = \mathrm{diag}(k_d^{(1)}, \ldots, k_d^{(n)})$ have positive diagonal entries.

Bent_beam {`nep,real`}. This 6×6 nonlinear eigenvalue problem arises in the study of a bent beam model [22]. The matrix-valued function takes the form

$$F(\lambda) = \begin{bmatrix} f_-(\lambda) & g_-(\lambda) & -f_+(\lambda) & -g_+(\lambda) & 0 & 0 \\ 0 & 0 & \alpha(\lambda) g_-(\lambda) & \alpha(\lambda) f_+(\lambda) & \sin\beta(\lambda)\ell & 0 \\ g_-(\lambda) & f_+(\lambda) & g_+(\lambda) & f_-(\lambda) & 0 & 0 \\ 0 & 0 & \alpha(\lambda)^2 EI f_-(\lambda) & \alpha(\lambda)^2 EI g_-(\lambda) & -\beta(\lambda)\tau\cos\beta(\lambda)\ell & 0 \\ \alpha(\lambda)^2 EI f_+(\lambda) & \alpha(\lambda)^2 EI g_+(\lambda) & 0 & 0 & 0 & \beta(\lambda)\tau\sin\beta(\lambda)\ell \\ \alpha(\lambda) g_+(\lambda) & \alpha(\lambda) f_-(\lambda) & 0 & 0 & 0 & -\cos\beta(\lambda)\ell \end{bmatrix},$$

where

$$\begin{aligned} f_+(\lambda) &= \cosh\alpha(\lambda)\ell + \cos\alpha(\lambda)\ell, & f_-(\lambda) &= \cosh\alpha(\lambda)\ell - \cos\alpha(\lambda)\ell, \\ g_+(\lambda) &= \sinh\alpha(\lambda)\ell + \sin\alpha(\lambda)\ell, & g_-(\lambda) &= \sinh\alpha(\lambda)\ell - \sin\alpha(\lambda)\ell, \end{aligned}$$

$$\alpha(\lambda) = \sqrt{\lambda}(m/EI)^{1/4}, \qquad \beta(\lambda) = \lambda\sqrt{m/\tau},$$

$\ell = 23.5$ in is the length of the beam, $EI = 38.92 \times 10^3$ lb in^2, $\tau = 4.93 \times 10^4$ lb in^2/rad, and $m = 1.833e - 4$ lb sec^2/in^2.

The eigenvalues of F correspond to the frequencies for flexure torsion oscillations of the bent beam.

Bucking_plate {`nep,real,symmetric`}. This is a 3×3 nonlinear eigenvalue problem that arises from a small bucking plate model. The symmetric matrix-valued function describing the nonlinear eigenproblem is given by

$$F(\lambda) = \begin{bmatrix} \dfrac{\lambda(1 - 2\lambda\cot 2\lambda)}{\tan\lambda - \lambda} + 10 & \dfrac{\lambda(2\lambda - \sin 2\lambda)}{\sin 2\lambda(\tan\lambda - \lambda)} & 2 \\ \dfrac{\lambda(2\lambda - \sin 2\lambda)}{\sin 2\lambda(\tan\lambda - \lambda)} & \dfrac{\lambda(1 - 2\lambda\cot 2\lambda)}{\tan\lambda - \lambda} + 4 & 2 \\ 2 & 2 & 8 \end{bmatrix}.$$

The eigenvalues of interest are the ones with smaller moduli. This nonlinear eigenvalue problem has been suggested by Melina Freitag through a private communication.

Canyon_particle {`nep,parameter_dependent,sparse,scalable,banded,low-rank`}. This nonlinear eigenproblem comes from a finite-element approximation of the Schrödinger equation for a

Table 4: The first half of the problems in NLEVP.

Name	Description
acoustic_wave_1d	Acoustic wave problem in 1 dimension.
acoustic_wave_2d	Acoustic wave problem in 2 dimensions.
bcc_traffic	QEP from stability analysis of chain of non-identical cars.
bent_beam	6-by-6 NEP from a bent beam model.
bicycle	2-by-2 QEP from the Whipple bicycle model.
bilby	5-by-5 QEP from bilby population model.
bucking_plate	3-by-3 NEP from a bucking plate model.
butterfly	Quartic matrix polynomial with T-even structure.
canyon_particle	NEP from the Schroedinger equation on a canyon-like shape.
cd_player	QEP from model of CD player.
circular_piston	Sparse QEP from model of circular piston.
closed_loop	2-by-2 QEP associated with closed-loop control system.
concrete	Sparse QEP from model of a concrete structure.
damped_beam	QEP from simply supported beam damped in the middle.
damped_gyro	QEP from a damped gyroscopic system.
deformed_consensus	n-by-n QEP from multi-agent systems theory.
dirac	QEP from Dirac operator.
disk_brake100	100-by-100 QEP from a disk brake model.
disk_brake4669	4669-by-4669 QEP from a disk brake model.
distributed_delay1	3-by-3 NEP from distributed delay system.
elastic_deform	QEP from elastic deformation of anisotropic material.
fiber	NEP from fiber optic design.
foundation	Sparse QEP from model of machine foundations.
gen_hyper2	Hyperbolic QEP constructed from prescribed eigenpairs.
gen_tantipal2	T-anti-palindromic QEP with eigenvalues on the unit circle.
gen_tpal2	T-palindromic QEP with prescribed eigenvalues on the unit circle.
gun	NEP from model of a radio-frequency gun cavity.
hadeler	NEP due to Hadeler.
hospital	QEP from model of Los Angeles Hospital building.
intersection	10-by-10 QEP from intersection of three surfaces.
loaded_string	REP from finite element model of a loaded vibrating string.
metal_strip	QEP related to stability of electronic model of metal strip.
mirror	Quartic PEP from calibration of cadioptric vision system.
mobile_manipulator	QEP from model of 2-dimensional 3-link mobile manipulator.
nep1	2-by-2 basic NEP example.
nep2	3-by-3 basic NEP example.
omnicam1	9-by-9 QEP from model of omnidirectional camera.

Table 5: The second half of the problems in NLEVP.

Name	Description
`omnicam2`	15-by-15 QEP from model of omnidirectional camera.
`orr_sommerfeld`	Quartic PEP arising from Orr-Sommerfeld equation.
`pdde_stability`	QEP from stability analysis of discretized PDDE.
`pdde_symmetric`	n-by-n NEP from a partial delay differential equation.
`photonic_crystal`	REP from dG-FEM of wave propagation in a periodic medium.
`pillbox_cavity`	170562-by-170562 NEP from a RF pillbox cavity.
`pillbox_small`	20-by-20 NEP from a RF pillbox cavity.
`planar_waveguide`	Quartic PEP from planar waveguide.
`plasma_drift`	Cubic PEP arising in Tokamak reactor design.
`power_plant`	8-by-8 QEP from simplified nuclear power plant problem.
`qep1`	3-by-3 QEP with known eigensystem.
`qep2`	3-by-3 QEP with known, nontrivial Jordan structure.
`qep3`	3-by-3 parametrized QEP with known eigensystem.
`qep4`	3-by-4 QEP with known, nontrivial Jordan structure.
`qep5`	3-by-3 nonregular QEP with known Smith form.
`railtrack`	QEP from study of vibration of rail tracks.
`railtrack_rep`	REP from study of vibration of rail tracks.
`railtrack2`	Palindromic QEP from model of rail tracks.
`railtrack2_rep`	REP from model of rail tracks.
`relative_pose_5pt`	Cubic PEP from relative pose problem in computer vision.
`relative_pose_6pt`	QEP from relative pose problem in computer vision.
`sandwich_beam`	NEP from model of a clamped sandwich beam.
`schrodinger`	QEP from Schrodinger operator.
`shaft`	QEP from model of a shaft on bearing supports with a damper.
`sign1`	QEP from rank-1 perturbation of sign operator.
`sign2`	QEP from rank-1 perturbation of 2*sin(x) + sign(x) operator.
`sleeper`	QEP modelling a railtrack resting on sleepers.
`speaker_box`	QEP from model of a speaker box.
`spring`	QEP from finite element model of damped mass-spring system.
`spring_dashpot`	QEP from model of spring/dashpot configuration.
`surveillance`	21-by-16 QEP from surveillance camera callibration.
`time_delay`	3-by-3 NEP from time-delay system.
`time_delay2`	2-by-2 NEP from a time-delay system.
`utrecht1331`	1331-by-1331 QEP from propagation of sound waves.
`wing`	3-by-3 QEP from analysis of oscillations of a wing in an airstream.
`wiresaw1`	Gyroscopic QEP from vibration analysis of a wiresaw.
`wiresaw2`	QEP from vibration analysis of wiresaw with viscous damping.

particle in a canyon-shaped 3 Ev potential well [15] and [30]. The corresponding matrix-valued function takes the form

$$F(\lambda) = H - \lambda I - \sum_{k=1}^{n_z} e^{\mathrm{i}\sqrt{m(\lambda - a_k)}} L_k U_k^T,$$

where H is a $n \times n$ sparse matrix, I is the identity matrix, and L_k, U_k are $n \times 2$ sparse matrices.

The user may set different parameters. They can change the mass of the particle m (default value is $m = 0.2$), the width of the canyon and of the valley (default values are $w_1 = 2$ and $w_2 = 2.2$), and the length of the canyon (default value is $l = 4$). The unit of measures are electron masses and nanometers, respectively. Finally, the user may also change the size s of the grid (default is $s = 0.05$). This influences n_z and the size n of the matrices. The formulas are

$$n_z = \lfloor 4/s \rfloor + 1, \qquad n = n_z(\lfloor 10/s \rfloor + 1),$$

so that for the default value of s, we have that $n_z = 81$ and $n = 16281$.

The eigenvalues of interest are the ones between a_1 and a_2, which are equal to $a_1 \approx -0.1920$ and $a_2 \approx -0.1320$ when default values are used. We refer to [30] for more information on the physics of the problem.

A part of the MATLAB file is an adaptation of `particle_init.m` by William Vandenberghe, which can be found in NLEIGS v0.5 [4] available at `http://twr.cs.kuleuven.be/research/software/nleps/nleigs.php`.

Circular_piston {`pep`,`qep`,`real`,`sparse`}. This is a quadratic matrix polynomial

$$Q(\lambda) = \lambda^2 M + \lambda E + K$$

arising in an axi-symmetric infinite element model of a circular piston [9]. The matrices are banded and have dimension 2025; M is real symmetric and singular; E and K are nonsymmetric.

Damped_gyro {`pep`,`qep`,`real`,`parameter-dependent`,`sparse`,`scalable`,`banded`}. This is a quadratic eigenvalue problem from a model of a damped gyroscopic system proposed in [20, Section 5.3]. It takes the form

$$Q(\lambda) = \lambda^2 M + \lambda(G + \epsilon D) + K,$$

where $M = M^T$, $K = K^T$, $D = D^T$ and $G = -G^T$ are real $n \times n$ matrices corresponding to the mass, stiffness, damping, and gyroscopic forces, respectively, and $\epsilon \geq 0$. The coefficient matrices are constructed as follows:

$$\begin{aligned} M &= c_{11} I \otimes \tilde{M} + c_{12} \tilde{M} \otimes I, \\ G &= c_{21} I \otimes \tilde{G} + c_{22} \tilde{G} \otimes I, \\ K &= c_{31} I \otimes \tilde{K} + c_{32} \tilde{K} \otimes I, \\ D &= c_{41} I \otimes \tilde{D} + c_{42} \tilde{D} \otimes I, \end{aligned}$$

where the c_{ij} are positive constants that are entries of $C \in \mathbb{R}^{4\times 2}$,

$$\tilde{M} = (4I + B + B^T)/6, \quad \tilde{G} = B - B^T, \quad \tilde{K} = B + B^T - 2I, \quad \tilde{D} = B + B^T + 2I,$$

and

$$B = \begin{bmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{m \times m}$$

is the shift-down matrix. The default values are $n = 36$ (i.e., $m = 6$), $\epsilon = 10^{-4}$ and

$$C = \begin{bmatrix} 1 & 1.3 \\ 0.1 & 1.1 \\ 1 & 1.2 \\ 1.05 & 0.9 \end{bmatrix}.$$

Deformed_consensus {`pep,qep,real,symmetric,parameter-dependent,scalable`}. This problem builds a $n \times n$ quadratic matrix polynomial, the deformed Laplacian, that arises in the study of the stability properties of the deformed consensus protocol in control theory [27]. It takes an $n \times n$ adjacency matrix A as an input and creates

$$F(\lambda) = I - A\lambda + (D - I)\lambda^2,$$

where I is the identity matrix and D is the degree matrix $D = \text{diag}(Ae)$, with e the vector or all ones. Problem provided by Fabio Morbidi.

Disk_brake100 {`qep,pep,parameter-dependent`}. This is a parameter-dependent quadratic eigenvalue problem of dimension $n = 100$ arising in the numerical simulation of disk brake squeal [12, Sec. 2.2]. It takes the form

$$Q(\lambda) = \lambda^2 M + D(\omega) + K,$$

where

$$D(\omega) = D_M + \frac{1}{\omega} D_R + \omega D_G, \qquad K = K_E + K_R.$$

The mass matrix M is symmetric and positive semidefinite, the material damping matrix D_M, the friction-induced damping matrix D_R and the elastic stiffness matrix K_E are all symmetric but the gyroscopic matrix D_G is skew-symmetric and the matrix K_R that describes circulatory effects is nonsymmetric. The real parameter $\omega \in [\omega_{\min}, \omega_{\max}]$ is the rotational speed of the brake disk. The default value is $\omega = 2\pi$. Matrices provided by Volker Mehrmann.

Disk_brake4669 {`qep,pep,parameter-dependent,sparse`}. This is a parameter-dependent quadratic eigenvalue problem of dimension $n = 4669$ arising in the numerical simulation of disk brake squeal [12, Sec. 2.3]. It takes the form

$$Q(\lambda) = \lambda^2 M + D(\omega) + K(\omega),$$

where

$$D(\omega) = D_M + \frac{1}{\omega} D_R + \omega D_G + \frac{1}{F_{\text{ref}}} D_4, \qquad K(\omega) = K_E + K_R + \omega^2 K_G.$$

As for the `disk_brake100` problem, the mass matrix M is symmetric and positive semidefinite, the material damping matrix D_M, the friction-induced damping matrix D_R, the elastic stiffness matrix K_E and the geometric stiffness matrix K_G are all symmetric. The gyroscopic matrix D_G is skew-symmetric and the matrix K_R describing circulatory effects is nonsymmetric. The matrix D_4 accounts for frequency dependent damping. The matrices were constructed for a reference frequency $F_{\text{ref}} = 1600$Hz. The real parameter $\omega \in [\omega_{\min}, \omega_{\max}]$ is the rotational speed of the brake disk. The default value is $\omega = 2\pi$. Matrices provided by Volker Mehrmann.

Distributed_delay1 {`nep,real,solution`}. Delay-differential equations involving a weighted mean of past states, also known as delay-differential equations with distributed delays, can be natural when modeling delays where the value of a delay is not accurately known. This nonlinear eigenvalue problem from [23] stems from a delay-differential equation with a distributed delay and has the form

$$R(\lambda) = -\lambda I + A_1 + A_2 e^{-\tau\lambda} + A_3 \int_{-1}^{0} e^{\lambda s} (e^{(s+\frac{1}{2})^2} - e^{\frac{1}{4}})\, ds.$$

where $A_1, A_2, A_3 \in \mathbb{R}^{3\times 3}$. Problem provided by Elias Jarlebring.

Elastic_deform {`pep,qep, real,T-even,parameter-dependent,sparse,scalable,banded`}. This is a quadratic eigenvalue problem from a model for the elastic deformation of anisotropic materials described in [20]. It takes the form

$$Q(\lambda) = \lambda^2 M + \lambda G + K,$$

where $M = M^T$, $K = K^T$, and $G = -G^T$ are the real $n \times n$ matrices defined in the `damped_gyro` problem. The default value is $n = 36$.

Nep1 {`nep`}. This 2×2 nonlinear matrix function $F(\lambda)$ has the form

$$F(\lambda) = \begin{bmatrix} e^{\mathrm{i}\lambda^2} & 1 \\ 1 & 1 \end{bmatrix}.$$

$F(\lambda)$ is singular when $e^{\mathrm{i}\lambda^2} = 1$, hence the eigenvalues are $\lambda_n = \{\sqrt{2\pi n}, \mathrm{i}\sqrt{2\pi n}\}$ for $n \in \mathbb{Z}$. Notice that every eigenvalue is simple, except for $\lambda_0 = 0$, which has algebraic multiplicity equals to two and geometric multiplicity to two–see [14] for more details.

Nep2 {`nep, real`}. This 3×3 nonlinear matrix function $F(z)$ has the form

$$F(z) = \begin{bmatrix} 2e^z + \cos z - 14 & (z^2-1)\sin z + (2e^z + 14)\cos z & 2e^z - 14 \\ (z+3)(e^z - 7) & \sin z + (z + 3(e^z - 7)\cos z & (z+3)(e^z-7) \\ e^z - 7 & (e^z - 7)\cos z & e^z - 7 \end{bmatrix}$$

and can be shown to be equivalent to

$$\begin{bmatrix} \cos z & & \\ & \sin z & \\ & & e^z - 7 \end{bmatrix},$$

therefore its eigenvalues are $\pi \log 7$ and $k\pi/2$ for every $k \in \mathbb{Z}$. This problem appeared in a 2008 presentation by Asakura, Sakurai and Tadano [2].

Pdde_symmetric {`nep, real, symmetric, scalable, sparse, banded`}. This is a nonlinear eigenvalue problem of scalable dimension n. It arises in the modelling of a partial delay differential equation (PDDE) with Dirichlet boundary condition

$$\begin{cases} u_t(x,t) &= \Delta u(x,t) + a(x)u(x,t) + b(x)u(x,t-2), \\ u(x,t) &= 0 \qquad x \in \partial\Omega, \quad t \geq 0, \end{cases}$$

defined on $\Omega = [0,\pi] \times [0,\pi]$ for $t \geq 0$, where $a(x) = \sin^2(x_1)\sin^2(x_2)$ and $b(x) = \sin(x_1 + x_2) + 1.31$. Using the standard 5-point stencil finite difference approximation to approximate the Laplacian operator on an $(m+1) \times (m+1)$ uniform grid, we have the algebraic eigenproblem of the form

$$T(\lambda) = (M + A) - \lambda I + e^{-2\lambda} B,$$

where the matrices $M, A, B \in \mathbb{R}^{n \times n}$ with $n = (m-1)^2$ discretize the Laplacian operator, $a(x)$ and $b(x)$, respectively. The default input is $m = 128$ (i.e.,$n = 16129$), for which the variational principle (see[14, Sec. 3]) is satisfied on $(-5.33, 0)$. The eigenvalues closest to zero are the ones of interest. Problem provided by Fei Xue.

Photonic_crystal {`rep, symmetric, sparse, parameter-dependent,scalable`}. This rational eigenvalue problem models the propagation of TM polarized waves in a 2D photonic crystal of periodically arranged unit squares. The nonlinearity is caused by the frequency dependency of the relative permittivity, which is modelled by a rational function, the so-called Lorentz model. The discretization of the governing Helmholtz equation has been performed with a high-order discontinuous Galerkin method, as discussed in [11]. The resulting eigenvalue problem takes the form

$$F(\lambda)x = \left[G - \lambda^2 \epsilon_0 M_0 - \lambda^2 \epsilon_1(\lambda) M_1\right] x = 0,$$

where G, M_0, M_1 are real symmetric $n \times n$ matrices with $n \in \{288, 720, 1344, 2160\}$. The matrices M_0 and M_1 are positive semidefinite. The coefficient ϵ_0 is constant, while $\epsilon_1(\lambda)$ takes the form

$$\epsilon_1(\lambda) = \epsilon_1 + \sum_{\ell=1}^{L} \frac{\lambda_{P,\ell}^2}{\lambda_{0,\ell}^2 - \lambda^2 - \mathrm{i}\gamma_\ell \lambda},$$

with parameters $\epsilon_1, \lambda_{P,\ell}, \lambda_{0,\ell}, \gamma_\ell$ for $\ell = 1, \ldots, L$. By default, $\epsilon_0 = 1$ and a synthetic Lorentz model with $L = 2$ terms is used for $\epsilon_1(\lambda)$. See, e.g., [10] for other examples. The eigenvalues of interest are usually those closest to the real axis, below the first pole of the Lorentz model. Problem supplied by Daniel Kressner and Cedric Effenberger.

Pillbox_cavity {`nep,real,symmetric,sparse,banded`}. This nonlinear eigenvalue problem is obtained by removing the cathode plate from the `gun` problem [29, p. 1039]. The eigenvalue problem is of the form

$$F(\lambda) = K - \lambda M + \mathrm{i}\sqrt{\lambda - k_1} W_1 + \mathrm{i}\frac{\lambda}{\sqrt{(\lambda - k_2)}} W_2 + \mathrm{i}\sqrt{\lambda - k_3} W_3 + \mathrm{i}\sqrt{\lambda - k_4} W_4 + \mathrm{i}\frac{\lambda}{\sqrt{(\lambda - k_4)}} W_5,$$

where $K, M, W_1, W_2, W_3, W_4, W_5$ are real symmetric matrices of size 170562×170562, with W_j, $j = 1, \dots, 5$ of low rank. The constants are

$$k_1 = 362.5201105315107, \quad k_2 = 477.9508476756487,$$
$$k_3 = 770.9410421638146, \quad k_4 = 1581.040692232823.$$

The eigenvalues of interest are the ones inside the interval $[1000, 2200]$ (i.e., $[1.0, 2.2]$ GHz). Problem supplied by Roel Van Beeumen.

Pillbox_small {`nep,symmetric`}. This nonlinear eigenvalue problem is a small variation of the `pillbox_cavity` problem of size 170562×170562 [29, p. 1039]. The eigenvalue problem has the form

$$F(\lambda) = K - \lambda M + \mathrm{i}\sqrt{\lambda - \sigma_1^2} W_1 + \mathrm{i}\sqrt{\lambda - \sigma_2^2} W_2$$

where K, M, W_1, W_2 are real, symmetric matrices of size 20×20. The user may use an optional input to call three different subproblems. The possible values are `a`, `b` and `brd`. The values of σ_i are

	`a`	`b`	`brd`
σ_1	0.15	1.2	1.2
σ_2	0.7	2.4	2.4

This problem can be found in another format in Osni Marques' GITHUB repository `https://github.com/oamarques/PAL`.

Railtrack_rep {`rep,sparse,low-rank`}. This is a rational matrix-valued function of size 1005,

$$R(\lambda) = \lambda A^T + B + \lambda^{-1} A$$

with $B = B^T$, corresponding to the original rational formulation of the `railtrack` problem already described in [7]. The matrices, the parameters and the references are described in `railtrack` and in [16], [17], [21], [25].

Railtrack2_rep {`rep,sparse,scalable,parameter-dependent,low-rank`}. This is a $n \times n$ complex rational matrix-valued function of the form

$$R(\lambda) = \lambda A^T(\omega) + B(\omega) + \lambda^{-1} A(\omega),$$

corresponding to the rational formulation of the quadratic eigenproblem associated with the `railtrack2` problem. The matrices $A(\omega), B(\omega) = B^T(\omega)$, which are are $m \times m$ block matrices with 705×705 blocks so that $n = 705m$, dependent quadratically on the real parameter ω—see [8], [13], [19]. Default values are $m = 51$ (so that $n = 35955$) and $\omega = 1000$.

Sandwich_beam {`nep,sparse,banded`}. This nonlinear eigenvalue problem arises from the vibration analysis of a clamped sandwich beam with viscoelastic core [1]. The problem is of the form

$$F(\lambda)x = \left(K_e - \lambda^2 M + \frac{\gamma_0 + \gamma_\infty (i\lambda\tau)^\alpha}{1 + (i\lambda\tau)^\alpha} K_v \right) x = 0,$$

where K_e, M, and K_v are $n \times n$ constant matrices with $n \in \{168, 840, 3360\}$, $\gamma_0 = 350.4$ kPa and $\gamma_\infty = 3.062$ MPa define the static and asymptotic shear modulus of the sandwich beam, τ is the relaxation time and α is the fractional parameter. The default values are $\tau = 8.230$ ns and $\alpha = 0.675$. The smallest eigenvalues in modulus are the ones of interest [5].

Time_delay2 {`nep,real,parameter-dependent`}. This 2×2 nonlinear matrix function has the form $F(\lambda) = \lambda I_2 + B_0 + A_1 e^{-\tau\lambda}$ with

$$B_0 = \begin{bmatrix} 5 & -1 \\ -2 & 6 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix},$$

and is the characteristic equation of the time-delay system

$$x'(t) = B_0 x(t) + A_1 x(t - \tau).$$

The default value of τ is 1. More information about this problem can be found in [24, Example 13] and [26, Sec. 2.4.2].

Utrecht1331 {`pep, qep, sparse, banded`}. This problem has the form

$$F(\lambda) = \lambda^2 A + \lambda B + C,$$

where B is a complex, non-Hermitian singular matrix, while A, C are real, symmetric positive definite of size 1331×1331. Despite the relatively small size, this is often a difficult problem for iterative methods. This QEP is a scaled–down model of the propagation of sound waves in room with a sound–absorbing material on a wall: see, for example, [3, Ex. 4], [18, Ex. 5.2] and [28].

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